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ESTIMATION OF STIFFNESS AND DISPLACEMENTS BY THE ANALYTICAL EXPRESS METHOD DURING THE OPERATION OF "SLIDING SHAFT-BEARING" CONNECTIONS.

Formulation of the problem.

Of all, when calculating structures in which a cantilever beam is used as an elastic element, it is not correct to solve this equation for the finite section by successive approximation of the transcendental equation. The exact solution of the differential equation leads to the transcendental equation in elliptic integrals of the 1st kind. From this equation, it is possible to determine the angle of rotation of the final section θk by the method of successive approximations, that is, in fact, by selection. For practical purposes, for example, when designing elastic elements of variable stiffness, it is necessary to have an explicit formula for the dependence of vertical displacement on the applied force. We differentiate the dependence of the reduced vertical displacement of the beam's end on the reduced load. Then we will take the inverse value from it. As a result, we get a graph of the reduced stiffness, which fits almost perfectly into the square parabola $y = 3 + {}^{x2}$. The internal energy of deformation of the previously unloaded and undeformed cantilever beam was determined. In order to verify this mathematical model, a comparison of the results of the nonlinear calculation with the results of the finite element calculations based on the traditional model and the one proposed in the work was performed. It was established that formulas (1) - (4) can be used when the displacement of the rod element does not exceed 80% of its length, which is a very significant geometric nonlinearity. The size of the stiffness matrix in the finite element method (FEM) is determined by the degree of discretization of the system. In practical tasks, the number of elements in the stiffness matrix is calculated by hundreds of thousands or millions. It is always a highly sparse matrix with a large number of zero elements.

Purpose of the study. The method of expert assessment of stiffness and displacement of the system is presented. For these purposes, an analytical express method is proposed. The necessary dependencies are presented in a sufficient amount, which indicate the problem of geometrically nonlinear bending, its connection with the basic differential equation of the elastic line of the beam. As an example, a cantilever beam with a cross-section of constant length, loaded at the end with a bending force, is considered. The equation of the elastic line's shape is shown for it.

Presentation of research material

Estimation of stiffness and displacements by the analytical express method during the operation of "Sliding shaft-bearing" connections

In design practice, the shipbuilder often has to face the need for an approximate expert assessment of system stiffness and movements. For these purposes, an analytical express method is proposed. Formulas that show all dependencies in an explicit form are obtained:

$$K_{\Phi} = 3 + \Phi^2; \quad (1)$$

$$v = \frac{1}{\sqrt{3}} arctg \frac{\Phi}{\sqrt{3}}; \quad (2)$$

$$K = \frac{3EJ}{L^3} + \frac{F^2 L}{EJ} = K_0 + \frac{F^2 L}{EJ}; \quad (3)$$

$$v = \frac{L}{\sqrt{3}} arctg \frac{FL^2}{\sqrt{3}EJ}. \quad (4)$$

where Φ = FL^2 /(EJ) – reduced force, or force coefficient of the form; V = v_k / L - reduced displacement, or geometric form factor;

K_{Φ} = KL^3 /(EJ) - reduced stiffness, or form stiffness factor.

The problem of geometrically nonlinear bending is also called the problem of large deflections or large displacements of beam cross sections in the literature.

Geometrically nonlinear bending is subject to the basic differential equation of an elastic line:

$$EJ \frac{d\theta}{ds} = M, \quad (5)$$

where EJ is the bending stiffness of the beam; M - external bending moment load in this section; s is a coordinate measured along the elastic beam lines; c - angle of rotation of the section. At the same time, curvature

$$\frac{d\theta}{ds}$$

cannot be replaced by the second derivative of the deflection $\frac{d^2 v}{dx^2}$, as in the linear case.

The basic relationship connecting the curvature with the bending moment was first obtained by Jacob Bernoulli. Then this task was dealt with by L. Euler and J. L. Lagrange, who considered a cantilever beam with a load on the free end [1 - 3].

Consider a cantilever beam with a cross-section of constant length, loaded at the end with a bending force (Fig. 1).

The equation of the shape of the elastic line for a cantilever beam loaded at the end with a bending force, with linear elasticity of the material and high tensile stiffness, has the form

$$\frac{d\theta}{ds}=\frac{F\left(x_k-x\right)}{EJ}$$

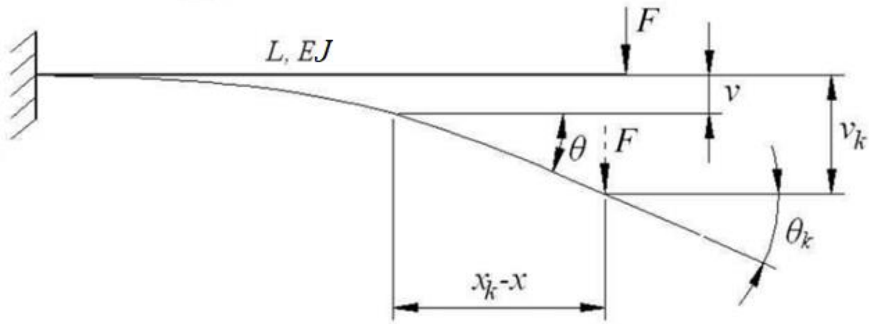

Fig. 1. Bending of the cantilever beam

For a cantilever beam with a cross-section of constant length, loaded at the end by a bending force, equation (5) is written as follows:

$$\frac{d\theta}{ds}=\frac{F\left(x_k-x\right)}{EJ}. \qquad (6)$$

where x is the horizontal coordinate of the current section; x_k - horizontal coordinate of the final section; F - applied load.

For a finite cross section, this equation is solved by successive approximation of the transcendental equation

$$\int_{\varphi}^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\sqrt{\frac{FL^2}{EJ}}, \qquad (7)$$

where

$$k=\sqrt{\frac{1+\sin\theta_k}{2}};\varphi=\arcsin\frac{1}{k\sqrt{2}}.$$

In most cases, this formula is linearized [40 - 42], considering

$$\frac{d\theta}{ds}=\frac{d^2v}{ds^2};x_k=L. \qquad (8)$$

where L is the length of the beam, v is the vertical displacement of the section.

However, there are a number of tasks (primarily when calculating structures in which a cantilever beam is used as an elastic element) where this assumption is unacceptable.

The exact solution of the differential equation (6) leads to the transcendental equation in elliptic integrals of the 1st kind (7). From this equation, it is possible to determine the angle of rotation of the final section θk by the method of successive approximations, i.e., in fact, by selection. The vertical and horizontal movement of the end of the beam are determined by formulas:

$$\frac{v_k}{L}=1-\sqrt{\frac{4EJ}{FL^2}}\cdot\int_{\varphi}^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt;$$

$$\frac{u_k}{L}=1-\sqrt{\frac{2EJ\sin\theta_k}{FL^2}} \qquad (9)$$

For practical purposes, for example, when designing elastic elements of variable stiffness, it is necessary to have an explicit formula for the dependence of vertical displacement on the applied force.

Let's write Hooke's law in differential form:

K • dv_k = dF,

where dv_k - load increment and displacement of the end of the rod, respectively, K is the coefficient of proportionality between these values (stiffness), depending, generally speaking, on the shape of the rod.

If the dependence of the rod’s stiffness on the shape is expressed in terms of displacement, formula (10) will be written

K (v_k) dv = dF

or

$$K\left(v_k\right)=\frac{dF}{dv_k}, \qquad (11)$$

and if in the form of a function of an external force, then

$$K\left(F\right)=1/\frac{dv_k}{dF}. \qquad (12)$$

Let's rewrite Hooke's law in dimensionless quantities. Then

$$K_\Phi dV = d\Phi. \tag{13}$$

$\Phi = \dfrac{FL^2}{EJ}$ - reduced force, or force coefficient of the form

$V = \dfrac{v_k}{L}$ -reduced displacement, or geometric form factor;

$K_\Phi = \dfrac{KL^2}{EJ}$ - reduced stiffness, or form stiffness factor.

For the linear case, given the relation

$$vL_k = \frac{FL^3}{3EJ}, \tag{14}$$

a stiffer form factor can be written as:

$$K_\Phi = \frac{\Phi}{v} = \frac{FL^3}{EJ} / \frac{FL^3}{3EJ \cdot L} = 3. \tag{15}$$

We numerically differentiate the dependence of the reduced vertical displacement of the end of the beam on the reduced load. Then, in accordance with expression (12), we will take the reciprocal of it. The resulting reduced stiffness graph is shown in Fig. 1. It is easy to see that it fits almost perfectly into the square a parabola $y = 3 + x^2$.

Suppose that the value

K $_\phi$ = 3 + Φ^2 *(16)*

is a solution to equation (13).

Then, integrating it, we get the total displacement:

$$\int_{V_1}^{V_2} dV = \int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{K_\Phi},$$

$$V_2 - V_1 = \int_{\Phi_1}^{\Phi_2} \frac{d\Phi}{3+\Phi^2} = \frac{1}{\sqrt{3}}\left(arctg\frac{\Phi_2}{\sqrt{3}} - arctg\frac{\Phi_1}{\sqrt{3}} \right). \tag{17}$$

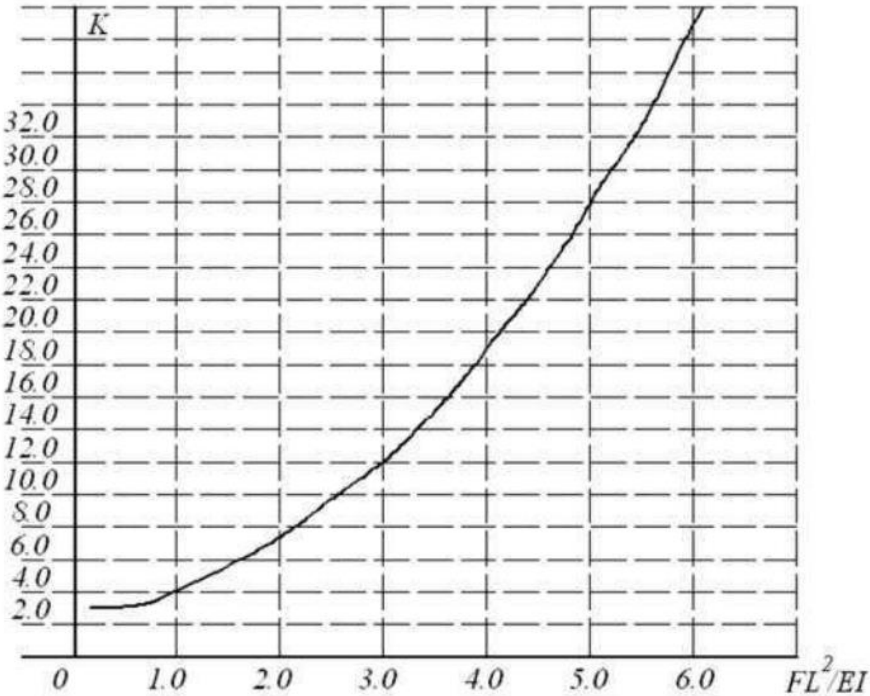

Fig. 1. The stiffness of the cantilever beam is given

For a pre-loaded and undeformed rod

$$V = \frac{1}{\sqrt{3}} arctg \frac{\Phi}{\sqrt{3}}. \tag{18}$$

The results of calculations according to formula (18) and their comparison with the values obtained according to formula (9) are given in the table. 1.

The calculation error by formula (18) in relation to the exact solution (9) does not exceed 0.4%.

Table 1. Comparative analysis of calculation methods

Φ	V	
	according to the formula (9)	according to the formula (18)
0.5	0.162	0.1623
1	0.302	0.3023
2	0.494	0.4948
3	0.603	0.6046
4	0.670	0.6710
5	0.714	0.7144
6	0.744	0.7446
7	0.767	0.7669
8	0.785	0.7838
9	0.799	0.7971
10	0.811	0.8079

Using the formulas obtained above, we determine the internal energy of deformation of a previously unloaded and undeformed cantilever beam. In a conservative system

$$U = A = \int_0^V F dv_k = \frac{EJ}{L}\int_0^V \Phi dV. \qquad (19)$$

Substituting (16) into (13), we write

$$dV = \frac{d\Phi}{K_\Phi} = \frac{d\Phi}{3+\Phi^2},$$

$$U = \frac{EJ}{L}\int_0^\Phi \frac{\Phi d\Phi}{3+\Phi^2} = \frac{1}{2}\frac{EJ}{L}\left[\ln\left(3+\Phi^2\right) - \ln 3\right] = \frac{1}{2}\frac{EJ}{L}\ln\left(1+\frac{\Phi^2}{3}\right). \qquad (20)$$

$$\Phi = \sqrt{3}tg\sqrt{3V}$$

Expressing from (18) and substituting it into (20), we obtain an expression for the strain energy due to displacement:

$$U = \frac{1}{2}\frac{EJ}{L}\ln\left(1 + tg^2\sqrt{3}V\right). \quad (21)$$

The strain energy calculated by this formula, for example, at V = 0.8, turns out to be 76% more than the linear formula gives.
Returning from the given values to the original values, we will rewrite formulas (6) - (8) and (20) - (21) for them:

$$K = \frac{3EJ}{L^3} + \frac{F^2L}{EJ} = K_0 + \frac{F^2L}{EJ};$$

$$v_k = \frac{L}{\sqrt{3}} arctg \frac{FL^2}{\sqrt{3}EJ};$$

$$U = \frac{1}{2}\frac{EJ}{L}\ln\left(1 + \frac{F^2L^4}{3E^2J^2}\right) = \frac{1}{2}\frac{EJ}{L}\ln\left(1 + tg^2\sqrt{3}\frac{v_k}{L}\right). \quad (22)$$

Thus, formulas (16) - (22) give all dependencies in an explicit form. However, these formulas have their own scope. This is indicated by the fact that the geometric coefficient of the form V as the load approaches infinity Φ →∞ converges to the value

$$v_{\max} = \frac{1}{\sqrt{3}} arctg\infty = \frac{\pi}{2\sqrt{3}} \approx 0{,}907.$$

and not to 1 as one would expect. Therefore, if the form's force coefficient exceeds 10, which corresponds to the transverse movement of the end of the beam more than 80% of the length of the beam (V = 0.8), it is not recommended to use the obtained formulas.
In fig. 2 the graphs of reduced stiffness KF (solid line), internal strain energy U (dotted line) and reduced load Φ (dashed line) depending on the geometric coefficient of the form V [1 - graphs of values according to formulas (14) - (18); 2 - graphs of values calculated according to the usual formulas of resistance of materials] are shown.

Formulas (16) - (22) are derived for the calculation scheme, the initial state of which is a rectilinear cantilever beam, the geometric and force coefficients of its shape are equal to zero. Now let's assume that a system whose shape corresponds to the force coefficient Φ0 (and the geometric

V0) is taken as the initial one, but at the same time there is no real load (Fig. 3).

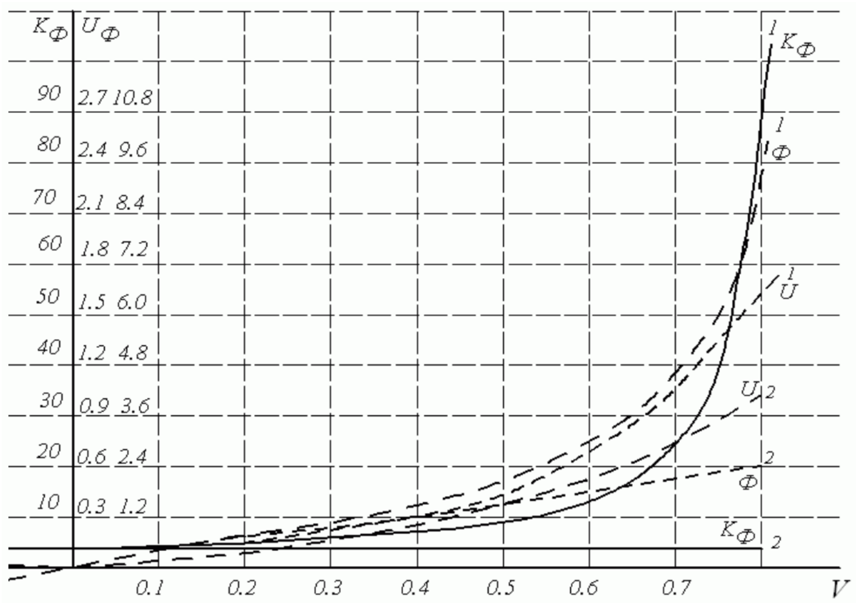

Fig. 2. Comparison of linear and non-linear calculations

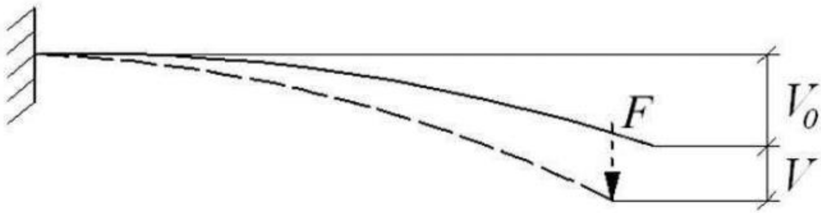

Fig. 3. Pre-bent cantilever beam

It is obvious that formulas (16) - (22) will also be fulfilled in this case, but they should take into account the initial conditions:

$$\Phi = \sqrt{3} tg \sqrt{3} V_0;$$

$$\Phi = \frac{FL^2}{EJ} + \Phi_0. \qquad (23)$$

The formulas for the transition of system 3 from the F1 position to the F2 position will have the following form

$$\Delta V = \frac{1}{\sqrt{3}}\left(arctg\frac{\Phi_2}{\sqrt{3}} - arctg\frac{\Phi_1}{\sqrt{3}} \right),$$

$$\Delta U = \frac{1}{2}\frac{EJ}{L}\ln\frac{3+\Phi_2^2}{3+\Phi_1^2} = \frac{1}{2}\frac{EJ}{L}\ln\frac{K_{\Phi_2}}{K_{\Phi_1}}. \quad (24)$$

In order to verify this mathematical model, a comparison of the results of nonlinear calculation according to formulas (1) - (4) with the results of finite element calculations based on the traditional model and the one proposed in the work was performed. It was established that formulas (1) - (4) can be used when the displacement of the rod element does not exceed 80% of its length, which is a very significant geometric nonlinearity.

Conclusions

1. The size of the stiffness matrix in the finite element method (FEM) is determined by the degree of discretization of the system. In practical tasks, the number of elements in the stiffness matrix is calculated by hundreds of thousands or millions. It is always a highly sparse matrix with a large number of zero elements.

2. One significant problem arises when solving nonlinear problems using the finite element method. If the stiffness matrix depends on the movements of the object, then the solution obtained with the help of MCE will be very different from the true one, and in some cases, it may even go beyond the scope of the problem definition. Therefore, it is necessary to apply an iterative approach, gradually increasing the external load from zero to a given value and recalculating the stiffness matrix at each step.

3. The proposed mathematical model of the dependence of stiffness, energy intensity and displacement of the elastic-dissipative ship beam system on the load and the analytical express method can be used at the design stage for an approximate expert assessment of stiffness and displacements, if the displacement of the rod element does not exceed 80% of its length.

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